Derivative of a system of implicit equations

The first equation represents the identity of aggregate income, where Y is income, C is consumption, I_0 is exogenous investment, and G_0 is exogenous government spending:

$$Y = C + I_0 + G_0$$

The consumption function depends on basic needs (α) and disposable income:

$$C = \alpha + \beta \cdot (Y - T)^{1/2}$$

where $0 < \beta < 1$

Private taxes T consist of a subsidy (τ) and a proportional tax (φ) on income:

$$T = \tau + \varphi Y$$

with $\tau < 0$ and $0 < \varphi < 1$

Endogenous variables: Y, C, T **Exogenous variables:** $I_0, G_0, \alpha, \beta, \tau, \varphi$

It is assumed that all parameters satisfy:

$$I_0 > 0, \quad G_0 > 0, \quad \alpha > 0, \quad 0 < \beta < 1, \quad \tau < 0, \quad 0 < \varphi < 1, \quad Y > 1$$

In addition:

$$0 < \frac{\beta(1-\varphi)}{2\sqrt{(1-\varphi)Y - \tau}} < 1$$

1. What is the effect of an increase in G_0 on consumption?
2. What is the effect of an increase in the marginal propensity to consume β on income Y? *Hint: insert T from the third equation into the second.*

Solution

1)

After substituting

$$T = \tau + \varphi Y$$

into the consumption function,

$$C = \alpha + \beta\sqrt{Y - (\tau + \varphi Y)} = \alpha + \beta\sqrt{(1-\varphi)Y - \tau}$$

the system reduces to two equations in Y and C:

$$Y - C - I_0 - G_0 = 0 \tag{1}$$

$$C - \alpha - \beta\sqrt{(1-\varphi)Y - \tau} = 0 \tag{2}$$

We define

$$\gamma = \frac{\beta\,(1-\varphi)}{2\,\sqrt{(1-\varphi)Y - \tau}}$$

We differentiate (1) and (2) implicitly with respect to G_0 (keeping $I_0, \alpha, \beta, \tau, \varphi$ constant):

$$\begin{cases} \dfrac{dY}{dG_0} - \dfrac{dC}{dG_0} = 1 \\ -\gamma\,\dfrac{dY}{dG_0} + \dfrac{dC}{dG_0} = 0 \end{cases}$$

In matrix form:

$$\begin{pmatrix} 1 & -1 \\ -\gamma & 1 \end{pmatrix} \begin{pmatrix} \dfrac{dY}{dG_0} \\ \dfrac{dC}{dG_0} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let A be the coefficient matrix and A_2 the matrix obtained by replacing the second column of A with $(1, 0)^T$. Then,

$$\det A = \begin{vmatrix} 1 & -1 \\ -\gamma & 1 \end{vmatrix} = 1 - \gamma, \quad \det A_2 = \begin{vmatrix} 1 & 1 \\ -\gamma & 0 \end{vmatrix} = \gamma$$

By Cramer's rule,

$$\frac{dC}{dG_0} = \frac{\det A_2}{\det A} = \frac{\gamma}{1-\gamma}$$

Since $0 < \gamma < 1$, we conclude:

$$\frac{dC}{dG_0} = \frac{\gamma}{1-\gamma} > 0$$

that is, an increase in G_0 raises consumption C

2)

After substituting

$$T = \tau + \varphi Y$$

into the consumption function,

$$C = \alpha + \beta\sqrt{(1-\varphi)Y - \tau}$$

the system becomes:

$$Y - C - I_0 - G_0 = 0 \tag{3}$$

$$C - \alpha - \beta\sqrt{(1-\varphi)Y - \tau} = 0 \tag{4}$$

Let

$$S = \sqrt{(1-\varphi)Y - \tau}, \qquad \delta = \frac{1-\varphi}{2S}$$

We differentiate (3) and (4) with respect to β (keeping $I_0, G_0, \alpha, \tau, \varphi$ constant):

$$\begin{cases} \dfrac{\partial Y}{\partial \beta} - \dfrac{\partial C}{\partial \beta} = 0 \\ -\beta\,\delta\,\dfrac{\partial Y}{\partial \beta} + \dfrac{\partial C}{\partial \beta} = S \end{cases}$$

In matrix form:

$$\begin{pmatrix} 1 & -1 \\ -\beta\,\delta & 1 \end{pmatrix} \begin{pmatrix} Y_\beta \\ C_\beta \end{pmatrix} = \begin{pmatrix} 0 \\ S \end{pmatrix}$$

The determinant of the coefficient matrix is:

$$\Delta = \det \begin{pmatrix} 1 & -1 \\ -\beta\delta & 1 \end{pmatrix} = 1 - \beta\,\delta$$

By Cramer's rule,

$$\frac{\partial Y}{\partial \beta} = \frac{1}{\Delta} \begin{vmatrix} 0 & -1 \\ S & 1 \end{vmatrix} = \frac{S}{1-\beta\,\delta} = \frac{\sqrt{(1-\varphi)Y - \tau}}{1 - \beta\,\dfrac{1-\varphi}{2\sqrt{(1-\varphi)Y-\tau}}}$$

$$\frac{\partial Y}{\partial \beta} = \frac{\sqrt{(1-\varphi)Y - \tau}}{1 - \dfrac{\beta\,(1-\varphi)}{2\sqrt{(1-\varphi)Y-\tau}}} > 0$$

since $\sqrt{(1-\varphi)Y - \tau} > 0$ and $1 - \beta\,\delta > 0$